A novel algorithm for G/D/1 queues

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Queuing theory deals with problems which involve queuing (or waiting). Typically we can talk of this individual sub-system as dealing with ***customers*** queuing for ***service***.
To analyse this sub-system we need information relating to:
- **arrival process**
- **service mechanism**
- **queue characteristics**

In order to get answers there are *two* basic approaches:
- analytic methods or queuing theory (formula based)
- simulation (computer based)

There is a standard notation system to classify queueing systems as A/B/C, where:
- A represents the probability distribution for the arrival process
- B represents the probability distribution for the service process
- C represents the number of channels (servers)

In our case
- D stands for a deterministic or constant value
- G stands for a general distribution (but with a known mean and variance)

1 Introduction

Many models of complex systems model the effects of noise as Markovian, such as Poisson or Wiener processes.
However, various studies have indicated that in many cases the underlying process may depart from Markovian behaviour and exhibit memory effects or "bursty" behaviour.

Non-Markovian stochastic processes are known for being difficult to solve analytically. However, there has been recent work on extending the Montroll-Weiss equations from anomalous transport to non-Markovian random walks on networks. This was then extended to model neuronal populations subject to noisy input with Gamma-distributed inter-event intervals.

2 Research Overview

In this project we imagine a queue with deterministic service times to be similar to an ODE with trivial dynamics.

This is done by transforming a discrete variable N (= number of customers) into a continuous variable T (= current total service time assuming no new inputs).

This continuous variable is then discretized according to the desired accuracy. Given a discrete realization of a probability distribution on this state space at a time t – $P(T,t)$ – we can compute $P(T, t+dt)$ in the absence of input as a simple shift in probability.

Meanwhile, stochastic input can be treated as a random walk on the discretized state space of T; for Poisson processes, this leads to a Master Equation.
To verify this, we need to use some Python code that successfully uses this method to reproduce results for M/D/1 queues [3].

For non-Poisson processes this Master Equation is replaced by a generalized version with a convolution with the memory kernel of the process. In [2] processes were used for which the memory kernel could be written down explicitly, while in this project we will extend it to more general processes, for which it will be computed using transform methods. We will then verify the accuracy of this method by comparing it to direct (Monte Carlo) simulations of queues.

This would be a significant improvement over current techniques for the modelling of such queues, which often involve significant approximations and/or specific strategies dependent on the type of non-Markov process being studied.

3 Methods

- Monte Carlo Simulations were fully realized using Python (numpy for distributions and matplotlib.pyplot for plots).
- ODEs were solved in Python using Euler's method for numerical approximation.
- Numerical Inverse Laplace Transform was done in Matlab using the classical Talbot method [4].

4 Material

I. Monte Carlo Simulation for G/D/1 queues.
In G/D/1 queues each random number represents an arrival of a customer. We are not interested in observing one queue, but in multiple queues at once. This gives us an understanding of averages and estimate probabilities of being in different states. For example these graphs show the estimate probabilities of being in a queue of 40 people with a fixed service time speed = 1.

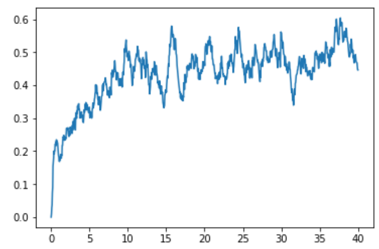

Fig 1. A queue with k = 2

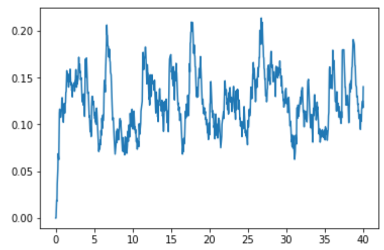

Fig 2. A queue with k = 5

We can get more graphs to observe out of these i.e. an average queue occupancy over a time.

II. A probability distribution function for a Markov chain can be computed by solving $\frac{dP}{dt} = M * P$, where P is a vector such that $P_i(t)$ is the probability of being in state i at time t, and M is the transition matrix.

If we have a 1-d matrix $P[:,t]$ for each time-step we can assume that

$$\frac{dP}{dt} = M * P[:,t]$$

Using the Euler's method for a numerical approximation we find

$$P[:,t+dt] = \mathrm{P}[:,\mathrm{dt}] + \mathrm{d}t\frac{dP}{dt}$$

Fig 3. is a plot of probabilities of different queue occupancy against time.
Each colour represents a different number of people in the queue.
For example, the dark blue curve is the probability that the queue is empty - it which starts at 1 and decreases to close to zero probability as people arrive.

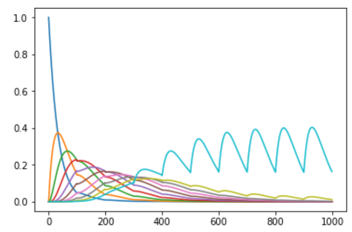

Fig 3.

III. We can also solve this equation analytically using Laplace Transform.
Some rearrangements yield to this equation:

$$\mathcal{L}\left\{\frac{dP}{dt}\right\} = (M - I)\hat{K}\hat{P}(s)$$

$$\text{Thus, } \frac{dP}{dt} = (M - I)[K(t) * P(t)]$$

For a Gamma distribution of shape α and rate v, K is the Laplace inverse of:

$$\hat{K} = \frac{sv^{\alpha}}{(s+v)^{\alpha} - v^{\alpha}}$$

Using this formula and Numerical Laplace Transform we can find functions for different α and v.

5 Future Directions

Possible future directions include:
1. Multi-server queues
2. Error analysis of the algorithm
3. Parallels with fractional differential equations
4. Applications to systems outside queuing theory

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[4] A Unified Framework for Numerically Inverting Laplace Transforms. Abate, Joseph, and Ward Whitt. INFORMS Journal of Computing, vol. 18.4 (2006): 408-421
[5] J E Beasley. 2018. Queuing theory. [ONLINE] Available at: http://people.brunel.ac.uk/~mastjjb/jeb/or/queue.html. [Accessed 8 October 2018].